ON GENERALIZED STATISTICS AND INTERACTIONS [1]

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Abstract

The concept of exchange braid statistics is generalized. The cross statistics is studied as a result of interaction. An algebraic model of a system of particles equipped with such statistics is described. The corresponding Fock space representation is also given.

1 . INTRODUCTION

It is well known that the standard approach to statistics of identical particles is based on the notion of the usual symmetric group S_n. This group describes the interchange process of indistinguishable particles. The transposition $(1,2) \longrightarrow (2,1)$ corresponding for the exchanging of two identical particles is represented by the map

$$\tau : x^1 \otimes x^2 \longrightarrow \pm\, x^2 \otimes x^1. \tag{1.1}$$

Every transposition yields a phase factor equal to ± 1 ($+1$ for bosons and -1 for fermions). If we replace the factor ± 1 by a complex parameter q, then we obtain the most simple generalization of the usual concept of statistics, namely the well–known q–statistics [1, 2, 3, 4, 5]. The corresponding particles are said to be quons. If $q := \exp(i\varphi)$, where $0 \leq \varphi < 2\pi$ is the so–called statistics parameter, then the corresponding q–statistics is determined by the value of φ. Observe that for $\varphi = 0$ we have bosons, and for $\varphi = \pm\pi$ – fermions. For arbitrary $\varphi \in [0, 2\pi)$ we obtain anyons [6, 7].

There is an interesting concept of an exotic statistics in a low dimensional space based on the notion of the braid group B_n [8, 9]. In this concept the configuration

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space for the system of n–identical particles moving on a manifold $\mathcal{M}$ is given by the following relation

$$Q_n(\mathcal{M}) = \Big(\mathcal{M}^{\times n} - D\Big)/S_n,$$

where D is the subset of the Cartesian product $\mathcal{M}^{\times n}$ on which two or more particles occupy the same position. The group $\pi_1\left(Q_n(M)\right) \equiv B_n(M)$ is just the n–string braid group on $\mathcal{M}$. Note that the statistics of a system of particles is determined by the group Σ_n [8, 9]. This group is a subgroup of the braid group $B_n(\mathcal{M})$ corresponding for the interchange process of two arbitrary indistinguishable particles. It is an extension of the symmetric group S_n. The mathematical formalism related to the braid group has been developed intensively by Majid, see [10, 11, 12, 13, 14, 15, 16] for example. An algebraic formalism for a particle system with generalized statistics has been considered by the author [17, 18, 19, 20, 21]. It is interesting that in this algebraic approach all commutation relations for particles equipped with an arbitrary statistics can be described as a representation of the so–called quantum Weyl algebra $\mathcal{W}$ (or Wick algebra) [27, 28, 29, 30, 31]. In this attempt the creation and annihilation operators act on an algebra $\mathcal{A}$. The creation operators act as the multiplication in this algebra and the annihilation ones act as a noncommutative contraction (noncommutative partial derivatives). The algebra $\mathcal{A}$ play the role of noncommutative Fock space. The application for particles in singular magnetic field has been given by the author [24, 25, 26]. Note that similar approach has been also considered by others authors [32, 33, 34, 35, 36]. An interesting concept related to generalized statistics has been also given in [37, 38].

In this paper we are going to study of a system of charged particles moving under influence of an intermediate quantum field. Our fundamental assumption is that every charged particle is transform under interaction into a system consisting a charge and quanta of the field. Such system behaves like free particles moving in certain effective space. It is showed that the system is endowed with the so–called cross statistics. This statistics is a result of inetractions. For the description of such statistics we develop the algebraic model of a system with generalized statistics studied previously by the author [39, 40].

2 . FUNDAMENTAL ASSUMPTIONS

We are going here to study a system of charged particles with certain dynamical interaction. It is natural to expect that some new and specific quantum states of the system have appear as a result of interaction. We would like to describe all such states. In order to do this we assume that the interaction can be described by an intermediate quantum field. Our fundamental assumption is that every charged particle is transform under interaction into a system consisting a charge and N–species of quanta of the field. A system which contains a charge and certain number of quanta as a result of interaction with the quantum field is said to be a *dressed particle* [43]. Next we assume that every dressed particle is a composite object equipped with an internal structure. Obviously the structure is determined by the interaction with the quantum field. We describe

the structure of a dressed particle as a nonlocal system which contains n centers. Such centers behave like free particles moving on certain effective space. Every center is equipped with ability for absorption and emission of quanta of the intermediate field. A center dressed with a single quantum of the field is said to be a *quasiparticle*. Quasiparticles represent elementary excited quantum states of the given system and they are are described by as a finite set of elements

$$Q := \{x^i : i = 1, \ldots, N < \infty\} \tag{2.1}$$

which form a basis for a linear space E over a field of complex numbers $\mathbb{C}$. A center which is an empty place for a quantum is called a *quasihole*. Quasiholes represent conjugate states and they are corresponding to the basis

$$Q^* := \{x^{*i} : i = N, N-1, \ldots, 1\}. \tag{2.2}$$

for the complex conjugate space E^*. The pairing $g_E : E^* \otimes E \longrightarrow \mathbb{C}$ and the corresponding scalar product is given by

$$g_E(x^{*i} \otimes x^j) = \langle x^i | x^j \rangle := \delta^{ij}. \tag{2.3}$$

A center which contains any quantum is said to be *neutral*. It represents the ground state $|0>= \mathbf{1}$ of the system. A neutral center can be transform into a quasiparticle or a quasihole by an absorption or emission process of single quantum, respectively. Quasiparticles and quasiholes as components of certain dressed particle have also their own statistics. It is interesting that there is a statistics of new kind, namely a *cross statistics*. This statistics is determined by an exchange process of quasiholes and quasiparticles. Note that the exchange is not a real process but an effect of interaction. Such exchange means annihilation of a quasiparticle on certain place and simultaneous creation of quasihole on an another place. This statistics is described by an operator T called an *elementary cross* or *twist*. This operator is linear, invertible and Hermitian. It is given by its matrix elements $T : E^* \otimes E \longrightarrow E \otimes E^*$

$$T(x^{*i} \otimes x^j) = \Sigma\, T^{ij}_{kl} x^k \otimes x^{*l}. \tag{2.4}$$

The usual exchange statistics of quasiparticles is described a linear B satisfying the standard braid relations

$$B^{(1)} B^{(2)} B^{(1)} = B^{(2)} B^{(1)} B^{(2)}, \tag{2.5}$$

where $B^{(1)} := B \otimes id$ and $B^{(2)} := id \otimes B$. The exchange process determined by the operator B is a real process. Such exchange process is possible if the dimension of the effective space is equal or great than two. Hence in this case we need two operators T and B for the description of our system with generalized statistics. These operators are not arbitrary. They must satisfy the following consistency conditions

$$\begin{gathered} B^{(1)} T^{(2)} T^{(1)} = T^{(2)} T^{(1)} B^{(2)}, \\ (id_{E \otimes E} + \tilde{T})(id_{E \otimes E} - B) = 0, \end{gathered} \tag{2.6}$$

where the operator $\tilde{T} : E \otimes E \longrightarrow E \otimes E$ is given by its matrix elements

$$(\tilde{T})^{ij}_{kl} = T^{ki}_{lj}. \tag{2.7}$$

We need a solution of these conditions for the construction of an example of a system with generalized statistics. Note that the general solution for these conditions is not known. Hence we must restrict our attention to some particular cases only. One can use solutions used in noncommutative differential calculi in order to give some examples [41]. In the one–dimensional case there is no place for such exchange process. Hence in this case only the cross statistics is possible, the exchange braid statistics not exists.

3. HERMITIAN WICK ALGEBRAS

Let us consider a pair of unital and associative algebras $\mathcal{A}$ and $\mathcal{A}^*$. We assume that they are conjugated. This means that there is an antilinear and involutive isomorphism $(-)^* : \mathcal{A} \longrightarrow \mathcal{A}^*$ and we have the following relations

$$m_{\mathcal{A}^*}(b^* \otimes a^*) = (m_{\mathcal{A}}(a \otimes b))^*, \quad (a^*)^* = a, \tag{3.1}$$

where $a, b \in \mathcal{A}$ and a^*, b^* are their images under the isomorhism $(-)^*$. Both algebras $\mathcal{A}$ and $\mathcal{A}^*$ are graded

$$\mathcal{A} := \bigoplus_n \mathcal{A}^n, \quad \mathcal{A}^* := \bigoplus_n \mathcal{A}^{*n}. \tag{3.2}$$

A linear mapping $\Psi : \mathcal{A}^* \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}^*$ such that we have the following relations

$$\begin{array}{l} \Psi \circ (id_{\mathcal{A}^*} \otimes m_{\mathcal{A}}) = (m_{\mathcal{A}} \otimes id_{\mathcal{A}^*}) \circ (id_{\mathcal{A}} \otimes \Psi) \circ (\Psi \otimes id_{\mathcal{A}}), \\ \Psi \circ (m_{\mathcal{A}^*} \otimes id_{\mathcal{A}}) = (id_{\mathcal{A}} \otimes m_{\mathcal{A}^*}) \circ (\Psi \otimes id_{\mathcal{A}^*}) \circ (id_{\mathcal{A}^*} \otimes \Psi) \\ (\Psi(b^* \otimes a))^* = \Psi(a^* \otimes b) \end{array} \tag{3.3}$$

is said to be a cross symmetry or $*$–twist [42]. We use here the notation

$$\Psi(b^* \otimes a) = \Sigma a_{(1)} \otimes b^*_{(2)} \tag{3.4}$$

for $a \in \mathcal{A}, b^* \in \mathcal{A}^*$.

The tensor product $\mathcal{A} \otimes \mathcal{A}^*$ of algebras $\mathcal{A}$ and $\mathcal{A}^*$ equipped with the multiplication

$$m_\Psi := (m_{\mathcal{A}} \otimes m_{\mathcal{A}^*}) \circ (id_{\mathcal{A}} \otimes \Psi \otimes id_{\mathcal{A}^*}) \tag{3.5}$$

is an associative algebra called a Hermitian Wick algebra [27, 42] and it is denoted by $\mathcal{W} = \mathcal{W}_\Psi(\mathcal{A}) = \mathcal{A} \otimes_\Psi \mathcal{A}^*$. This means that the Hermitian Wick algebra $\mathcal{W}$ is the tensor cross product of algebras $\mathcal{A}$ and $\mathcal{A}^*$ with respect to the cross symmetry Ψ [42]. Let H be a linear space. We denote by $L(H)$ the algebra of linear operators acting on H. One can prove [42] that we have the

Theorem: Let $\mathcal{W} \equiv \mathcal{A} \otimes_\Psi \mathcal{A}^*$ be a Hermitian Wick algebra. If $\pi_{\mathcal{A}} : \mathcal{A} \longrightarrow L(H)$ is a representation of the algebra $\mathcal{A}$, such that we have the relation

$$\begin{array}{c} (\pi_{\mathcal{A}}(b))^* \pi_{\mathcal{A}}(a) = \Sigma \pi_{\mathcal{A}}(a_{(1)}) \pi_{\mathcal{A}^*}(b^*_{(2)}), \\ \pi_{\mathcal{A}^*}(a^*) := (\pi_{\mathcal{A}}(a))^*, \end{array} \tag{3.6}$$

then there is a representation $\pi_{\mathcal{W}} : \mathcal{W} \longrightarrow L(H)$ of the algebra $\mathcal{W}$. We use the following notation

$$\pi_{\mathcal{A}}(x^i) \equiv a^+_{x^i}, \quad \pi_{\mathcal{A}^*}(x^{*i}) \equiv a_{x^{*i}}. \tag{3.7}$$

The relations (3.6) are said to be *commutation relations* if there is a positive definite scalar product on H such that operators $a^+_{x^i}$ are adjoint to $a_{x^{*i}}$ and vice versa. Let us consider a Hermitian Wick algebra $\mathcal{W}$ corresponding for a system with generalized statistics. For the construction of such algebra we need a pair of algebras $\mathcal{A}$, $\mathcal{A}^*$ and a cross symmetry Ψ. It is natural to assume that these algebras have E and E^* as generating spaces, respectively, and there is the following condition for the cross symmetry

$$\Psi|_{E^*\otimes E} = T + g_E. \tag{3.8}$$

4. FOCK SPACE REPRESENTATION

Let us consider the Fock space representation of the algebra $\mathcal{W}$ corresponding for a system with generalized statistics. For the ground state and annihilation operators we assume that

$$\langle 0|0\rangle = 0, \quad a_{s^*}|0\rangle = 0 \quad \text{for} \quad s^* \in \mathcal{A}^*. \tag{4.1}$$

In this case the representation act on the algebra $\mathcal{A}$. Creation operators are defined as a multiplication in the algebra $\mathcal{A}$

$$a^+_s t := m_{\mathcal{A}}(s \otimes t), \quad \text{for} \quad s, t \in \mathcal{A}. \tag{4.2}$$

The proper definition of the action of annihilation operators on the whole algebra $\mathcal{A}$ is a problem.

If the action of annihilation operators are given in such a way that there is unique, nondegenerate, positive definite scalar product on $\mathcal{A}$, creation operators are adjoint to annihilation ones and vice versa, then we say that we have a well–defined system with generalized statistics in the Fock representation [40].

Let us consider some examples for such systems. Assume that quasiparticles and quasiholes are moving on one dimensional effective space. In this case the algebra of states $\mathcal{A}$ is the full tensor algebra TE over the space E, and the conjugate algebra $\mathcal{A}^*$ is identical with the tensor algebra TE^*. If $T \equiv 0$ then we obtain the most simple example of well–defined system with generalized statistics. The corresponding statistics is the so–called infinite (Bolzman) statistics [1, 2, 40]. If $T : E^* \otimes E \longrightarrow E \otimes E^*$ is an arbitrary nontrivial cross operator, then there is the cross symmetry $\Psi^T : TE^* \otimes TE \longrightarrow TE \otimes E^*$. It is defined by a set of mappings $\Psi_{k,l} : E^{*\otimes k} \otimes E^{*\otimes k} \longrightarrow E^{\otimes l} \otimes E^{*\otimes k}$, where $\Psi_{1,1} \equiv R := T + g_E$, and

$$\begin{aligned} \Psi_{1,l} &:= R^{(l)}_l \circ \ldots \circ R^{(1)}_l, \\ \Psi_{k,l} &:= (\Psi_{1,l})^{(1)} \circ \ldots \circ (\Psi_{1,l})^{(k)}, \end{aligned} \tag{4.3}$$

here $R_l^{(i)} : E_l^{(i)} \longrightarrow E_l^{(i+1)}$, $E_l^{(i)} := E \otimes \ldots \otimes E^* \otimes E \otimes \ldots \otimes E$ ($l+1$-factors, E^* on the i-th place, $i \leq l$) is given by the relation

$$R_l^{(i)} := \underbrace{id_E \otimes \ldots \otimes R \otimes \ldots \otimes id_E}_{l \ times},$$

where R is on the i-th place, $(\Psi_{1,l})^{(i)}$ is defined in similar way like $R^{(i)}$. The commutation relations (3.6) can be given here in the following form

$$a_{x^{*i}} a^+_{x^j} - T^{ij}_{kl} \ a^+_{x^k} a_{x^{*l}} = \delta^{ij} \mathbf{1}. \tag{4.4}$$

If the linear operator $\tilde{T} : E \otimes E \longrightarrow E \otimes E$ with the following matrix elements

$$(\tilde{T})^{ij}_{kl} = T^{ki}_{lj}. \tag{4.5}$$

is bounded, we have the following Yang-Baxter equation on $E \otimes E \otimes E$

$$(\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) = (id_E \otimes \tilde{T}) \circ (\tilde{T} \otimes id_E) \circ (id_E \otimes \tilde{T}), \tag{4.6}$$

and $||\tilde{T}|| \leq 1$, then according to Bożejko and Speicher [5] there is a positive definite scalar product. Note that the existence of a nontrivial kernel of the operator $P_2 \equiv id_{E\otimes E} + \tilde{T}$ is essential for the nondegeneracy of the scalar product [27]. One can see that if this kernel is trivial, then we obtain well–defined system with generalized statistics [28, 30].

If the dimension of the effective space is great than one and the kernel of P_2 is nontrivial, then the scalar product is degenerate. Hence we must remove this degeneracy by factoring the mentioned above scalar product by the kernel. In this case we have $\mathcal{A} := TE/I$, $\mathcal{A}^* := TE^*/I^*$, where $I := gen\{id_{E\otimes E} - B\}$ is an ideal in TE and $B : E \otimes E \longrightarrow E \otimes E$ is a linear and invertible operator satisfying the braid relation (2.5) and the consistency conditions (2.6), I^* is the corresponding conjugated ideal in TE^*. One can see that there is the cross symmetry and the action of annihilation operators can be defined in such a way that we obtain the well–defined system with the usual braid statistics [28, 30]. We have here the following commutation relations

$$\begin{array}{l} a_{x^{*i}} a^+_{x^j} - T^{ij}_{kl} \ a^+_{x^k} a_{x^{*l}} = \delta^{ij} \mathbf{1} \\ a_{x^{*i}} a_{x^{*j}} - B^{kl}_{ij} \ a_{x^{*l}} a_{x^{*k}} = 0, \\ a^+_{x^i} a^+_{x^j} - B^{ij}_{kl} \ a^+_{x^k} a^+_{x^l} = 0. \end{array} \tag{4.7}$$

Observe that for $T \equiv B \equiv \tau$, where τ represents the transposition (1.1) we obtain the usual canonical (anti–)commutation relations for bosons or fermions.

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References

[1] O. W. Greenberg, *Phys. Rev. Lett.* **64** 705 (1990).

[2] O. W. Greenberg, *Phys. Rev.* **D 43**, 4111 (1991).

[3] R. N. Mohapatra, *Phys. Lett.* **B 242**, 407 (1990).

[4] S. Meljanac and A. Perica, Mod. Phys. Lett. **A9** 3293 (1994).

[5] M. Bożejko, R. Speicher, *Math. Ann.* **300**, 97, (1994).

[6] F. Wilczek, Phys. Rev. Lett. **48** 114 (1982).

[7] F. Wilczek and A. Zee, Phys. Rev. Lett. **51** 2250 (1983).

[8] Y.S. Wu, J.Math.Phys. **52**, 2103, 1984

[9] T.D. Imbo and J. March–Russel, Phys. Lett.**B252**, 84, 1990

[10] S. Majid, *Int. J. Mod. Phys.***A5**, 1 (1990).

[11] S. Majid, *J. Math. Phys.***34**, 1176, (1993).

[12] S. Majid, *J. Math. Phys.***34**, 4843, (1993).

[13] S. Majid, *J. Math. Phys.***34**, 2045 (1993).

[14] S. Majid, Algebras and Hopf Algebras in Braided Categories, in Advanced in Hopf Algebras, Plenum 1993.

[15] S. Majid, *J. Geom. Phys.* **13**, 169 (1994).

[16] S. Majid, AMS Cont. Math. **134**, 219 (1992).

[17] W. Marcinek, *J. Math. Phys.* **33**, 1631 (1992).

[18] W. Marcinek, Rep. Math. Phys. 34, 325 (1994).

[19] W. Marcinek,*Rep. Math. Phys. 33*, 117, (1993).

[20] W. Marcinek, *J. Math. Phys.* **35**, 2633, (1994).

[21] W. Marcinek, *Int. J. Mod. Phys.* **A10**, 1465-1481 (1995).

[22] W. Marcinek, On the deformation of commutation relations, in Proceedings of the XIII Workshop in Geometric Methods in Physics, July 1-7, 1994 Białowieza, Poland, ed. J. Antoine, Plenum Press 1995.

[23] W. Marcinek, On quantum Weyl algebras and generalized quons, in Proceedings of the symposium: Quantum Groups and Quantum Spaces, Warsaw, November 20-29, 1995, Poland, ed. by R. Budzynski, W. Pusz and S. Zakrzewski, Banach Center Publications, Warsaw 1997.

[24] W. Marcinek, On algebraic model of composite fermions and bosons, in Proceedings of the IXth Max Born Symposium, Karpacz, September 25 - September 28, 1996, Poland, ed by J. Lukierski et al Polish Scientific Publishers, Warszawa 1997.

[25] W. Marcinek, Topology and quantization, in Proceedings of the IVth International School on Theoretical Physics, Symmetry and Structural Properties, Zajaczkowo k. Poznania, August 29 - September 4 1996, Poland, ed. by B. Lulek, T. Lulek and R. Florek, p. 415–124, World Scientific, Singapore 1997 and hep-th/97 05 098, May 1997.

[26] W. Marcinek, Particles in singular magnetic fields, q-alg/97060031, June 1997, J. Math. Phys.**39**, 818–830 (1998).

[27] P.E.T. Jorgensen, L.M. Schmith, and R.F. Werner, Positive representation of general commutation relations allowing Wick ordering, *J. Funct. Anal.* **134**, 33 (1995).

[28] W. Marcinek and R. Rałowski, Particle operators from braided geometry, in "Quantum Groups, Formalism and Applications" XXX Karpacz Winter School in Theoretical Physics, 1994, Eds. J. Lukierski et al., 149-154 (1995).

[29] W. Marcinek and Robert Rałowski, On Wick Algebras with Braid Relations, *Preprint IFT UWr* **876/9**, (1994) and *J. Math. Phys.* **36**, 2803, (1995).

[30] R. Ralowski, On deformations of commutation relation algebras, *Preprint IFT UWr* **890**, (1995), q-alg/9506004.

[31] A. Borowiec, V. K. Kharchenko, Z. Oziewicz, Calculi on Clifford-Weyl and exterior algebras for Hecke braiding, Conferencia dictata por Zbigniew Oziewicz at Centro de Investigation en Matematicas, Guanajuato, Mexico, Seminario de Geometria, jueves 22 de abril de 1993 and at the Conference on Differential Geometric Methods, Ixtapa, Mexico, September 1993.

[32] R. Scipioni, *Phys. Lett.* **B327**, 56 (1994).

[33] Yu Ting and Wu Zhao-Yan, *Science in China* **A37**, 1472 (1994).

[34] S. Meljanac and A. Perica *Mod. Phys. Lett.* **A9**, 3293 (1994).

[35] M. Pillin, *Commun. Math. Phys.* **180**, 23 (1996).

[36] A. K. Mishra and G. Rajasekaran, *J. Math. Phys.* **38**, 466 (1997).

[37] G. Fiore and P. Schup, Statistics and Quantum Group Symmetries, in Proceedings of the symposium: Quantum Groups and Quantum Spaces, Warsaw, November 20-29, 1995, Poland, ed. by R. Budzynski, W. Pusz and S. Zakrzewski, Banach Center Publications, Warsaw 1997.

[38] S. Meljanac and M. Molekovic, Int. J. Mod. Phys. Lett.**A11**, 139 (1996).

[39] W. Marcinek, Categories and quantum statistics, in Proceedings of the symposium: Quantum Groups and their Applications in Physics, Poznan October 17-20, 1995, Poland, Rep. Math. Phys. **38**, 149-179 (1996)

[40] W. Marcinek, Remarks on Quantum Statistics, in Proceedings of the Conference "Particles, Fields and Gravitation", April 15 - 19, (1998), Lodz, Poland, ed. by J. Rembielinski, World Scientific, Singapore 1998, and math.QA./9806158.

[41] W. Marcinek, On Commutation Relations for Quons, Rep. Math. Phys. 41, 155 (1998).

[42] A. Borowiec and W. Marcinek, On twisted product of algebras, in preparation.

[43] W. Marcinek, On generalized statistics and one dimensional systems, in Proceedings of the III International Seminar "Hidden Symmetry", Rzeszow, October 20-22, 1998, Poland.